

Beyond Max-Cut: λ -Extendible Properties Parameterized Above the Poljak-Turzík Bound

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Abstract

Poljak and Turzík (Discrete Math. 1986) introduced the notion of λ -extendible properties of graphs as a generalization of the property of being bipartite. They showed that for any $0 < \lambda < 1$ and λ -extendible property Π , any connected graph G on n vertices and m edges contains a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges. The property of being bipartite is λ -extendible for $\lambda = 1/2$, and thus the Poljak-Turzík bound generalizes the well-known Edwards-Erdős bound for MAX-CUT.

We define a variant, namely *strong* λ -extendibility, to which the Poljak-Turzík bound applies. For a strongly λ -extendible graph property Π , we define the parameterized ABOVE POLJAK-TURZÍK (Π) problem as follows: Given a connected graph G on n vertices and m edges and an integer parameter k , does there exist a spanning subgraph H of G such that $H \in \Pi$ and H has at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges? The parameter is k , the surplus over the number of edges guaranteed by the Poljak-Turzík bound.

We consider properties Π for which the ABOVE POLJAK-TURZÍK (Π) problem is fixed-parameter tractable (FPT) on graphs which are $O(k)$ vertices away from being a graph in which each block is a clique. We show that for all such properties, ABOVE POLJAK-TURZÍK (Π) is FPT for all $0 < \lambda < 1$. Our results hold for properties of oriented graphs and graphs with edge labels.

Our results generalize the recent result of Crowston et al. (ICALP 2012) on MAX-CUT parameterized above the Edwards-Erdős bound, and yield FPT algorithms for several graph problems parameterized above lower bounds. For instance, we get that the above-guarantee MAX q -COLORABLE SUBGRAPH problem is FPT. Our results also imply that the parameterized above-guarantee ORIENTED MAX ACYCLIC DIGRAPH problem is FPT, thus solving an open question of Raman and Saurabh (Theor. Comput. Sci. 2006).

Keywords and phrases Algorithms and data structures; fixed-parameter algorithms; bipartite graphs; acyclic graphs.

1 Introduction

A number of interesting graph problems can be phrased as follows: Given a graph G as input, find a subgraph H of G with the largest number of edges such that H satisfies a specified property Π . Prominent among these is the MAX-CUT problem, which asks for a *bipartite* subgraph with the maximum number of edges. A *cut* of a graph G is a partition of the vertex

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set of G into two parts, and the *size* of the cut is the number of edges which *cross the cut*; that is, those which have their end points in distinct parts of the partition.

MAX-CUT

Input: A graph G and an integer k .

Question: Does G have a cut of size at least k ?

The MAX-CUT problem is among Karp’s original list of 21 NP-complete problems [14], and it has been extensively investigated from the point of view of various algorithmic paradigms. Thus, for example, Goemans and Williamson showed [12] that the problem can be approximated in polynomial-time within a multiplicative factor of roughly 0.878, and Khot et al. showed that this is the best possible assuming the Unique Games Conjecture [15].

Our focus in this work is on the *parameterized* complexity of a generalization of the MAX-CUT problem. The central idea in the parameterized complexity analysis [7, 11] of NP-hard problems is to associate a *parameter* k with each input instance of size n , and then to ask whether the resulting *parameterized problem* can be solved in time $f(k) \cdot n^c$ where c is a constant and f is some computable function. Parameterized problems which can be solved within such time bounds are said to be fixed-parameter tractable (FPT).

The *standard parameterization* of the MAX-CUT problem sets the parameter to be the size k of the cut being sought. This turns out to be not very interesting for the following reason: Let m be the number of edges in the input graph G . By an early result of Erdős [10], we know that every graph with m edges contains a cut of size at least $m/2$. Therefore, if $k \leq m/2$ then we can immediately answer YES. In the remaining case $k > m/2$, and there are less than $2k$ edges in the input graph. It follows from this bound on the size of the input that any algorithm—even a brute-force method—which solves the problem runs in FPT time on this instance.

The best lower bound known on the size of a largest cut for connected loop-less graphs on n vertices and m edges is $\frac{m}{2} + \frac{n-1}{4}$, as proved by Edwards [8, 9]. This is called the *Edwards-Erdős bound*, and it is the best possible in the sense that it is tight for an infinite family of graphs, for example, the class of cliques of odd order n . A more interesting parameterization of MAX-CUT is, therefore, the following:

MAX-CUT ABOVE TIGHT LOWER BOUND (MAX-CUT ATLB)

Input: A connected graph G , and an integer k .

Parameter: k

Question: Does G have a cut of size at least $\frac{m}{2} + \frac{n-1}{4} + k$?

In the work which introduced the notion of “above-guarantee” parameterization, Mahajan and Raman [16] showed that the problem of asking for a cut of size at least $\frac{m}{2} + k$ is FPT parameterized by k , and stated the fixed-parameter tractability of MAX-CUT ATLB as an open problem. This question was resolved quite recently by Crowston et al. [5], who showed that the problem is in fact FPT.

We generalize the result of Crowston et al. by extending it to apply to a special case of the so-called λ -*extendible properties*. Roughly stated¹, for a fixed $0 < \lambda < 1$ a graph property Π is said to be λ -extendible if: Given a graph $G = (V, E) \in \Pi$, an “extra” edge uv not in G , and any set F of “extra” edges each of which has one end point in $\{u, v\}$ and the other in V , there exists a graph $H \in \Pi$ which contains (i) all of G , (ii) the edge uv , and (iii) at

¹ See subsection 1.2 and section 2 for the definitions of various terms used in this section.

least a λ fraction of the edges in F . The notion was introduced by Poljak and Turzík who showed [18] that for any λ -extendible property Π and edge-weighting function $c : E \rightarrow \mathbb{R}^+$, any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ such that $c(F) \geq \lambda \cdot c(E) + \frac{1-\lambda}{2}c(T)$. Here $c(X)$ denotes the total weight of all the edges in X , and T is the set of edges in a minimum-weight spanning tree of G . It is not difficult to see that the property of being bipartite is λ -extendible for $\lambda = 1/2$, and so—once we assign unit weights to all edges—the Poljak and Turzík result implies the Edwards-Erdős bound. Other examples of λ -extendible properties—with different values of λ —include q -colorability and acyclicity in oriented graphs.

In this work we study the natural above-guarantee parameterized problem for λ -extendible properties Π , which is: given a connected graph $G = (V, E)$ and an integer k as input, does G contain a spanning subgraph $H = (V, F) \in \Pi$ such that $c(F) = \lambda \cdot c(E) + \frac{1-\lambda}{2}c(T) + k$? To derive a generic FPT algorithm for this class of problems, we use the “reduction” rules of Crowston et al. To make these rules work, however, we need to make a couple of concessions. Firstly, we slightly modify the notion of lambda extendibility; we define a (potentially) stronger notion which we name *strong λ -extendibility*. Every strongly λ -extendible property is also λ -extendible by definition, and so the Poljak-Turzík bound applies to strongly λ -extendible properties as well. Observe that for each way of assigning edge-weights, the Poljak and Turzík result yields a (potentially) different lower bound on the weight of the subgraph. Following the spirit of the question posed by Mahajan and Raman and solved by Crowston et al., we choose from among these the lower bound implied by the unit-edge-weighted case. This is our second simplification, and for this “unweighted” case the Poljak and Turzík result becomes: for any strongly λ -extendible property Π , any connected graph $G = (V, E)$ contains a spanning subgraph $H = (V, F) \in \Pi$ such that $|F| = \lambda \cdot |E| + \frac{1-\lambda}{2}(|V| - 1)$.

The central problem which we discuss in this work is thus the following; here $0 < \lambda < 1$, and Π is an arbitrary—but fixed—strongly λ -extendible property:

ABOVE POLJAK-TURZÍK (Π) (APT(Π))

Input: A connected graph $G = (V, E)$ and an integer k .

Parameter: k

Question: Is there a spanning subgraph $H = (V, F) \in \Pi$ of G such that $|F| \geq \lambda|E| + \frac{1-\lambda}{2}(|V| - 1) + k$?

1.1 Our Results and their Implications

We show that that the ABOVE POLJAK-TURZÍK (Π) problem is FPT for every strongly λ -extendible property Π for which APT(Π) is FPT on a class of “almost-forests of cliques”. Informally, this is a class of graphs which are a small number ($O(k)$) of vertices away from being a graph in which each block is a clique. This requirement is satisfied by the properties underlying a number of interesting problems, including MAX-CUT, MAX q -COLORABLE SUBGRAPH, and ORIENTED MAX ACYCLIC DIGRAPH.

The following is the main result of this paper.

► **Theorem 1.** *The ABOVE POLJAK-TURZÍK (Π) problem is fixed-parameter tractable for a λ -extendible property Π of graphs if*

- Π is strongly λ -extendible and
- ABOVE POLJAK-TURZÍK (Π) is FPT on almost-forests of cliques.

This also holds for such properties of oriented and/or labelled graphs.

We prove Theorem 1 using the classical “Win/Win” approach of parameterized complexity.

To wit: given an instance (G, k) of a strongly λ -extendible property Π , in polynomial time we either (i) show that (G, k) is a yes instance, or (ii) find a vertex subset S of G of size at most $6k/(1 - \lambda)$ such deleting S from G leaves a “forest of cliques”. To prove this we use the “reduction” rules used by Crowston et al [5] in the context of MAX-CUT.

Our main technical contribution is the proof that these rules are sufficient to show that *every* NO instance of $\text{APT}(\Pi)$ is at a vertex-deletion distance of $O(k)$ from a forest of cliques. This proof requires several new ideas: a result which holds for *all* strongly λ -extendible properties Π is a significant step forward from MAX-CUT. Our main result unifies and generalizes known results and implies new ones. Among these are MAX-CUT, finding a q -colorable subgraph of the maximum size, and finding a maximum-size acyclic subdigraph in an oriented graph. Using our theorem we also get a linear *vertex* kernel for maximum acyclic subdigraph, complementing the quadratic *arc* kernel by Gutin et al. [13].

Related Work

The notion of parameterizing above (or below) some kind of “guaranteed” values—lower and upper bounds, respectively—was introduced by Mahajan and Raman [16]. It has proven to be a fertile area of research, and MAX-CUT is now just one of a host of interesting problems for which we now have FPT results for such questions [19, 17, 13, 1, 3, 5, 4, 2].

1.2 Preliminaries

We use “graph” to denote simple graphs without self-loops, directions, or labels, and use standard graph terminology used by Diestel [6] for the terms which we do not explicitly define. We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of graph G , respectively. For $S \subseteq V(G)$, we use (i) $G[S]$ to denote the subgraph of G induced by the set S , (ii) $G \setminus S$ to denote $G[V(G) \setminus S]$, (iii) $\delta(S)$ to denote the set of edges in G which have exactly one end-point in S , and (iv) $e_G(S)$ to denote $|E(G[S])|$; we omit the subscript G if it is clear from the context. A *clique* in a graph G is a set of vertices C such that between any pair of vertices in C there is an edge in $E(G)$. A *block* of graph G is a maximal 2-connected subgraph of G and a graph G is a *forest of cliques*, if the vertices of each of its blocks form a clique. Thus a graph is a forest of cliques if and only if the vertex set of any cycle in the graph forms a clique. A *leaf clique* of a forest of cliques is a block of the graph, which corresponds to a leaf in its block forest. In other words, it is a block which contains at most one cut vertex of the graph.

For $F \subseteq E(G)$, (i) we use $G \setminus F$ to denote the graph $(V(G), E(G) \setminus F)$, and (ii) for a weight function $c : E(G) \rightarrow \mathbb{R}^+$, we use $c(F)$ to denote the sum of the weights of all the edges in F . A *graph property* is a collection of graphs. For $i, j \in \mathbb{N}$ we use K_i to denote the complete graph on i vertices, and $K_{i,j}$ to denote the complete bipartite graph in which the two parts of vertices are of sizes i, j .

Our results also apply to graphs with oriented edges, and those with edge labels. Subgraphs of an oriented or labelled graph G inherit the orientation or labelling—as is the case—of G in the natural manner: each surviving edge keeps the same orientation/labelling as it had in G . For a graph G of any kind, we use G_S to denote the simple graph obtained by removing all orientations and labels from G ; we say that G is connected (or contains a clique, and so forth) if G_S is connected (or contains a clique, and so forth).

2 Definitions

The following notion is a variation on the concept of λ -extendibility defined by Poljak and Turzík [18].

► **Definition 2** (Strong λ -extendibility). Let \mathcal{G} be the class of (possibly oriented and/or labelled) graphs, and let $0 < \lambda < 1$. A property $\Pi \subseteq \mathcal{G}$ is *strongly λ -extendible* if it satisfies the following:

Inclusiveness $\{G \in \mathcal{G} \mid G_S \in \{K_1, K_2\}\} \subseteq \Pi$

Block additivity $G \in \mathcal{G}$ belongs to Π if and only if each block of G belongs to Π .

Strong λ -subgraph extension Let $G \in \mathcal{G}$ and $S \subseteq V(G)$ be such that $G[S] \in \Pi$ and $G \setminus S \in \Pi$. For any weight function $c : E(G) \rightarrow \mathbb{R}^+$ there exists an $F \subseteq \delta(S)$ with $c(F) \geq \lambda \cdot c(\delta(S))$, such that $G \setminus (\delta(S) \setminus F) \in \Pi$.

The strong λ -subgraph extension requirement can be rephrased as follows: Let $V(G) = X \uplus Y$ be a cut of graph G such that $G[X], G[Y] \in \Pi$, and let F be the set of edges which cross the cut. For *any* weight function $c : F \rightarrow \mathbb{R}^+$, there exists a subset $F' \subseteq F$ such that (i) $c(F') \leq (1 - \lambda) \cdot c(F)$, and (ii) $(G \setminus F') \in \Pi$. Informally, one can pick a λ -fraction of the cut and delete the rest to obtain a graph which belongs to Π .

We recover Poljak and Turzík's definition of λ -extendibility from the above definition by replacing strong λ -subgraph extension with the following property:

λ -edge extension Let $G \in \mathcal{G}$ and $S \subseteq V(G)$ be such that $G_S[S]$ is isomorphic to K_2 and $G \setminus S \in \Pi$. For any weight function $c : E(G) \rightarrow \mathbb{R}^+$ there exists an $F \subseteq \delta(S)$ with $c(F) \geq \lambda \cdot c(\delta(S))$, such that $G \setminus (\delta(S) \setminus F) \in \Pi$.

Observe from the definitions that any graph property which is strongly λ -extendible is also λ -extendible. It follows that Poljak and Turzík's result for λ -extendible properties applies also to strongly λ -extendible properties.

► **Theorem 3** (Poljak-Turzík bound). [18] *Let \mathcal{G} be a class of (possibly oriented and/or labelled) graphs. Let $0 < \lambda < 1$, and let $\Pi \subseteq \mathcal{G}$ be a strongly λ -extendible property. For any connected graph $G \in \mathcal{G}$ and weight function $c : E(G) \rightarrow \mathbb{R}^+$, there exists a spanning subgraph $H \in \Pi$ of G such that $c(E(H)) \geq \lambda \cdot c(E(G)) + \frac{1-\lambda}{2}c(T)$, where T is the set of edges in a minimum-weight spanning tree of G_S .*

When all edges are assigned weight 1, we get:

► **Corollary 4.** *Let $\mathcal{G}, \lambda, \Pi$ be as in Theorem 3. Any connected graph $G \in \mathcal{G}$ on n vertices and m edges has a spanning subgraph $H \in \Pi$ with at least $\lambda m + \frac{1-\lambda}{2}(n-1)$ edges.*

Our results apply to properties which satisfy the additional requirement of being FPT on almost-forests of cliques.

► **Definition 5** (FPT on almost-forests of cliques). Let $0 < \lambda < 1$, and let Π be a strongly λ -extendible property (of graphs with or without orientations/labels). The STRUCTURED ABOVE POLJAK-TURZÍK (Π) problem is a variant of the ABOVE POLJAK-TURZÍK (Π) problem in which, along with the graph G and $k \in \mathbb{N}$, the input contains a set $S \subseteq V(G)$ such that $|S| = O(k)$ and $G \setminus S$ is a forest of cliques. We say that the property Π is *FPT on almost-forests of cliques* if the STRUCTURED ABOVE POLJAK-TURZÍK (Π) problem is FPT.

In other words, a λ -extendible property Π is FPT on almost-forests of cliques, if for any constant q there is an algorithm that, given a connected graph G, k and a set $S \subseteq V(G)$ of size at most $q \cdot k$ such that $G \setminus S$ is a forest of cliques, correctly decides whether (G, k) is a yes-instance of APT(Π) in $O(f(k) \cdot n^{O(1)})$ time, for some computable function f .

3 Fixed-Parameter Algorithms for Above Poljak-Turzík (II)

We now prove Theorem 1 using the approach which Crowston et al. used for MAX-CUT [5]. The crux of their approach is a polynomial-time procedure which takes the input (G, k) of MAX-CUT and finds a subset $S \subseteq V(G)$ such that (i) $G \setminus S$ is a forest of cliques, and (ii) if (G, k) is a NO instance, then $|S| \leq 3k$. Thus if $|S| > 3k$, then one can immediately answer YES; otherwise one solves the problem in FPT time using the fact that MAX-CUT is FPT on almost-forests of cliques (Definition 5).

The nontrivial part of our work consists of proving that the procedure for MAX-CUT applies also to the much more general family of strongly λ -extendible problems, where the bound on the size of S depends on λ . To do this, we show that each of the four rules used for MAX-CUT is safe to apply for any strongly λ -extendible property. From this we get

► **Lemma 6.** *Let $0 < \lambda < 1$, and let Π be a strongly λ -extendible graph property. Given a connected graph G with n vertices and m edges and an integer k , in polynomial time we can do one of the following:*

1. *Decide that there is a spanning subgraph $H \in \Pi$ of G with at least $\lambda m + \frac{1-\lambda}{2}(n-1) + k$ edges, or;*
 2. *Find a set S of at most $\frac{6}{1-\lambda}k$ vertices in G such that $G \setminus S$ is a forest of cliques.*
- This also holds for strongly λ -extendible properties of oriented and/or labelled graphs.*

We give an algorithmic proof of Lemma 6. Let (G, k) be an instance of ABOVE POLJAK-TURZÍK (II). The algorithm initially sets $\tilde{G} := G$, $\tilde{S} := \emptyset$, $\tilde{k} := k$, and then applies a series of rules to the tuple $(\tilde{G}, \tilde{S}, \tilde{k})$. Each application of a rule to $(\tilde{G}, \tilde{S}, \tilde{k})$ produces a tuple (G', S', k') such that (i) if $\tilde{G} \setminus \tilde{S}$ is connected then so is $G' \setminus S'$, and (ii) if $(\tilde{G} \setminus \tilde{S}, \tilde{k})$ is a NO instance of APT(Π) then so is $(G' \setminus S', k')$; the converse may not hold. The algorithm then sets $\tilde{G} := G'$, $\tilde{S} := S'$, $\tilde{k} := k'$, and repeats the process, till none of the rules applies. These rules—but for minor changes—and the general idea of “preserving a NO instance” are due to Crowston et al. [5].

We now state the four rules and show that they suffice to prove Lemma 6. We assume throughout that λ and Π are as in Lemma 6. For brevity we assume that the empty graph is in Π , and we let $\lambda' = \frac{1}{2}(1 - \lambda)$ so that $\lambda + 2\lambda' = 1$.

► **Rule 1.** Let $\tilde{G} \setminus \tilde{S}$ be connected. If $v \in (V(\tilde{G}) \setminus \tilde{S})$ and $X \subseteq (V(\tilde{G}) \setminus (\tilde{S} \cup \{v\}))$ are such that (i) $\tilde{G}[X]$ is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{v\})$, and (ii) $X \cup \{v\}$ is a clique in \tilde{G} , then delete X from \tilde{G} to get G' ; set $S' = \tilde{S}$, $k' = \tilde{k}$.

► **Rule 2.** Let $\tilde{G} \setminus \tilde{S}$ be connected. Suppose Rule 1 does not apply, and let X_1, \dots, X_ℓ be the connected components of $\tilde{G} \setminus (\tilde{S} \cup \{v\})$ for some $v \in V(\tilde{G}) \setminus \tilde{S}$. If at least one of the X_i s is a clique, and at most one of them is *not* a clique, then

- Delete all the X_i s which are cliques—let these be d in number—to get G' , and
- Set $S' := \tilde{S} \cup \{v\}$ and $k' := \tilde{k} - d\lambda'$.

► **Rule 3.** Let $\tilde{G} \setminus \tilde{S}$ be connected. If $a, b, c \in V(\tilde{G}) \setminus \tilde{S}$ are such that $\{a, b\}, \{b, c\} \in E(\tilde{G})$, $\{a, c\} \notin E(\tilde{G})$, and $\tilde{G} \setminus (\tilde{S} \cup \{a, b, c\})$ is connected, then

- Set $S' := \tilde{S} \cup \{a, b, c\}$ and $k' := \tilde{k} - \lambda'$.

► **Rule 4.** Let $\tilde{G} \setminus \tilde{S}$ be connected. Suppose Rule 3 does not apply, and let $x, y \in V(\tilde{G}) \setminus \tilde{S}$ be such that

1. $\{x, y\} \notin E(\tilde{G})$;

2. Let C_1, \dots, C_ℓ be the connected components of $\tilde{G} \setminus (\tilde{S} \cup \{x, y\})$. There is at least one C_i such that both $V(C_i) \cup \{x\}$ and $V(C_i) \cup \{y\}$ are cliques in $\tilde{G} \setminus \tilde{S}$, and there is at most one C_i for which this does *not* hold.

Then

- Delete all the C_i s which satisfy condition (2) to get G' , and,
- Set $S' := \tilde{S} \cup \{x, y\}$, $k' := \tilde{k} - \lambda'$.

Let (G^*, S, k^*) be the tuple which we get by applying these rules exhaustively to the input tuple (G, \emptyset, k) . To prove Lemma 6, it is sufficient to prove the following claims: (i) the rules can be exhaustively applied in polynomial time; (ii) $G \setminus S$ is a forest of cliques; (iii) the rules transform NO-instances to NO-instances; and, (iv) if (G, k) is a NO instance, then $|S| \leq q(\lambda)k$. We now proceed to prove these over several lemmata. Our rules are identical to those of Crowston et al. in how the rules modify the graph; the only difference is in how we change the parameter k . The first two claims thus follow directly from their work.

► **Lemma 7.** $[\star]^2$ *Rules 1 to 4 can be exhaustively applied to an instance (G, k) of ABOVE POLJAK-TURZÍK (II) in polynomial time. The resulting tuple (G^*, S, k^*) has $|V(G^*) \setminus S| \leq 1$.*

► **Lemma 8.** [5, Lemma 8] *Let (G^*, S, k^*) be the tuple obtained by applying Rules 1 to 4 exhaustively to an instance (G, k) of ABOVE POLJAK-TURZÍK (II). Then $G \setminus S$ is a forest of cliques.*

The correctness of the remaining two claims is a consequence of the λ -extendibility of property II, and we make critical use of this fact in building the rest of our proof. This is the one place where this work is significantly different from the work of Crowston et al.; they could take advantage of the special characteristics of *one specific* property, namely bipartitedness, to prove the analogous claims for MAX-CUT.

We say that a rule is *safe* if it preserves NO instances.

► **Definition 9.** Let $(\tilde{G}, \tilde{S}, \tilde{k})$ be an arbitrary tuple to which one of the rules 1, 2, 3, or 4 applies, and let (G', S', k') be the resulting tuple. We say that the rule is *safe* if, whenever $(G' \setminus S', k')$ is a YES instance of ABOVE POLJAK-TURZÍK (II), then so is $(\tilde{G} \setminus \tilde{S}, \tilde{k})$.

We now prove that each of the four rules is safe. For a graph G we use $val(G)$ to denote the maximum number of edges in a subgraph $H \in \Pi$ of G , and $pt(G)$ to denote the Poljak-Turzík bound for G . Thus if G is connected and has n vertices and m edges then $pt(G) = \lambda m + \lambda'(n - 1)$, and Corollary 4 can be written as $val(G) \geq pt(G)$. For each rule we assume that $G' \setminus S'$ has a spanning subgraph $H' \in \Pi$ with at least $pt(G' \setminus S') + k'$ edges, and show that $\tilde{G} \setminus \tilde{S}$ has a spanning subgraph $\tilde{H} \in \Pi$ with at least $pt(\tilde{G} \setminus \tilde{S}) + \tilde{k}$ edges.

We first derive a couple of lemmas which describe how contributions from subgraphs of a graph G add up to yield lower bounds on $val(G)$.

► **Lemma 10.** $[\star]$ *Let v be a cut vertex of a connected graph G , and let $\mathcal{C} = C_1, C_2, \dots, C_r$; $r \geq 2$ be sets of vertices of G such that for every $i \neq j$ we have $C_i \cap C_j = \{v\}$, there is no edge between $C_i \setminus \{v\}$ and $C_j \setminus \{v\}$ and $\bigcup_{1 \leq i \leq r} C_i = V(G)$. For $1 \leq i \leq r$, let $H_i \in \Pi$ be a subgraph of $G[C_i]$ with $pt(G[C_i]) + k_i$ edges, and let $H = (V(G), \bigcup_{i=1}^r E(H_i))$. Then H is a subgraph of G , $H \in \Pi$, and $|E(H)| \geq pt(G) + \sum_{i=1}^r k_i$.*

² Proofs of results marked with a \star have been moved to Appendix A.

► **Lemma 11.** $[\star]$ *Let G be a graph, and let $S \subseteq V(G)$ be such that there exists a subgraph $H_S \in \Pi$ of $G[S]$ with at least $pt(G[S]) + \lambda' + k_S$ edges, and a subgraph $\bar{H} \in \Pi$ of $G \setminus S$ with at least $pt(G \setminus S) + \lambda' + \bar{k}$ edges. Then there is a subgraph $H \in \Pi$ of G with at least $pt(G) + \lambda' + k_S + \bar{k}$ edges.*

This lemma has a useful special case which we state as a corollary:

► **Corollary 12.** *Let G be a graph, and let $S \subseteq V(G)$ be such that there exists a subgraph $H_S \in \Pi$ of $G[S]$ with at least $pt(G[S]) + \lambda' + k_S$ edges, and the subgraph $G \setminus S$ has a perfect matching. Then there is a subgraph $H \in \Pi$ of G with at least $pt(G) + \lambda' + k_S$ edges.*

Proof. Recall that the graph K_2 is in Π by definition, and observe that $pt(K_2) = \lambda + \lambda'$. Thus K_2 has $pt(K_2) + \lambda'$ edges. The corollary now follows by repeated application of Lemma 11, each time considering a new edge of the matching as the graph \bar{H} . ◀

The safeness of Rule 1 is now a consequence of the block additivity property.

► **Lemma 13.** *Rule 1 is safe.*

Proof. Let $C_1 = X \cup \{v\}$ and $C_2 = V(\tilde{G}) \setminus (\tilde{S} \cup X) = V(G') \setminus S'$. Observe that (i) v is a cut vertex of $\tilde{G} \setminus \tilde{S}$, $C_1 \cap C_2 = \{v\}$, there are no edges between $C_1 \setminus \{v\}$ and $C_2 \setminus \{v\}$ by assumptions of the rule, and $C_1 \cup C_2 = V(\tilde{G}) \setminus \tilde{S}$. Also by assumption, there is a spanning subgraph $H_1 \in \Pi$ of $G' \setminus S' = \tilde{G}[C_2]$ such that $|E(H_1)| \geq pt(G' \setminus S') + k'$. By Corollary 4 there is a subgraph $H_2 \in \Pi$ of $\tilde{G}[C_2]$ with $|E(H_2)| \geq pt(\tilde{G}[C_2])$. Hence Lemma 10 applies and $\tilde{G} \setminus \tilde{S}$ has a spanning subgraph $H \in \Pi$ with $E(H) \geq pt(\tilde{G} \setminus \tilde{S}) + k' = pt(\tilde{G} \setminus \tilde{S}) + \bar{k}$. ◀

We now prove some useful facts about certain simple graphs, in the context of strongly λ -extendible properties. Observe that every block of a forest is one of $\{K_1, K_2\}$, which are both in Π . From this and the block additivity property of Π we get

► **Observation 14.** Every forest (with every orientation and labeling) is in Π .

The graph $K_{2,1}$ is a useful special case.

► **Observation 15.** The graph $K_{2,1}$ —also with any kind of orientation or labelling—is in Π , and it has $pt(K_{2,1}) + \lambda' + \lambda'$ edges.

The graph obtained by removing one edge from K_4 is another useful object, since it always has more edges than its Poljak-Turzík bound.

► **Lemma 16.** $[\star]$ *Let G be a graph formed from the graph K_4 —also with any kind of orientation or labelling—by removing one edge. Then (i) $val(G) \geq 3$, (ii) $val(G) \geq 4$ if $\lambda > 1/3$, and (iii) $val(G) = 5$ if $\lambda > 1/2$. As a consequence,*

$$val(G) \geq pt(G) + \lambda' + \begin{cases} (1 - 3\lambda) & \text{if } \lambda \leq 1/3, \\ (2 - 3\lambda) & \text{if } 1/3 < \lambda \leq 1/2, \text{ and,} \\ (3 - 3\lambda) & \text{if } \lambda > 1/2. \end{cases} \quad (1)$$

The above lemmata help us prove that Rules 2 and 3 are safe.

► **Lemma 17.** $[\star]$ *Rule 2 is safe.*

Following the notation of Rule 3, observe that for the vertex subset $T = \{a, b, c\} \subseteq V(\tilde{G} \setminus \tilde{S})$ we have—from Observation 15—that $\tilde{G}[T] \in \Pi$ and $val(T) \geq pt(T) + \lambda' + \lambda'$. Since $G' \setminus S' = (\tilde{G} \setminus \tilde{S}) \setminus T$, if $val(G' \setminus S') \geq pt(G' \setminus S') + k'$ then applying Lemma 11 we get that $val(\tilde{G} \setminus \tilde{S}) \geq pt(\tilde{G} \setminus \tilde{S}) + \lambda' + k' = pt(\tilde{G} \setminus \tilde{S}) + \bar{k}$. Hence we get

► **Lemma 18.** *Rule 3 is safe.*

To show that Rule 4 is safe, we need a number of preliminary results. We first observe that—while the rule is stated in a general form—the rule only ever applies when it can delete exactly one component.

► **Observation 19.** $[\star]$ Whenever Rule 4 applies, there is exactly one component to be deleted, and this component has at least 2 vertices.

Our next few lemmas help us further restrict the structure of the subgraph to which Rule 4 applies. We start with a result culled from Crowston et al.’s analysis of the four rules.

► **Lemma 20.** $[5][\star]$ *If none of Rules 1, 2, and 3 applies to $(\tilde{G}, \tilde{S}, \tilde{k})$, and Rule 4 does apply, then one can find*

- *A vertex $r \in V(\tilde{G} \setminus \tilde{S})$ and a set $X \subseteq V(\tilde{G} \setminus \tilde{S})$ such that X is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{r\})$, and the graph $(\tilde{G} \setminus \tilde{S})[X \cup \{r\}]$ is 2-connected;*
- *Vertices $x, y \in X$ such that $\{x, y\} \notin E(\tilde{G})$ and*
 - *$(\tilde{G} \setminus \tilde{S}) \setminus \{x, y\}$ has exactly two components G', C ,*
 - *$r \in G'$; $C \cup \{x\}, C \cup \{y\}$ are cliques, and each of x, y is adjacent to some vertex in G'*

From this we get the following.

► **Lemma 21.** $[\star]$ *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if x, y are the vertices to be added to \tilde{S} and C the clique to be deleted, then $N(x) \cup N(y) \setminus (C \cup \tilde{S})$ contains at most one vertex z such that $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

We now show that in such a case $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{r\}$, and so the graph $\tilde{G} \setminus (\tilde{S} \cup \{r\})$ is not connected. First we need the following simple lemma.

► **Lemma 22.** $[\star]$ *Whenever Rule 4 applies, with x, y the vertices to be added to \tilde{S} and C the clique to be deleted, every u in $N(x) \setminus (C \cup \tilde{S})$ is a cut vertex in $\tilde{G} \setminus (\tilde{S} \cup \{x\})$ and every u in $N(y) \setminus (C \cup \tilde{S})$ is a cut vertex in $\tilde{G} \setminus (\tilde{S} \cup \{y\})$.*

This allows us to enforce a very special way of applying Rule 4.

► **Lemma 23.** $[\star]$ *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if x, y are the vertices to be added to \tilde{S} and C the clique to be deleted, then $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{z\}$, and $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

These lemmas help us prove that Rule 4 is safe.

► **Lemma 24.** $[\star]$ *Rule 4 is safe.*

The next lemma gives us a bound on the size of the set S which we compute.

► **Lemma 25.** $[\star]$ *Let \tilde{G} be a connected graph, $\tilde{S} \subseteq V(\tilde{G})$, and $\tilde{k} \in \mathbb{N}$, and let one application of Rule 1, 2, 3, or 4 to $(\tilde{G}, \tilde{S}, \tilde{k})$ result in the tuple (G', S', k') . Then $|S' \setminus \tilde{S}| \leq 3(\tilde{k} - k')/\lambda'$.*

Now we are ready to prove Lemma 6, and thence our main theorem.

Proof (of Lemma 6). Let (G, k) be an input instance of ABOVE POLJAK-TURZÍK (II), and let (G^*, S, k^*) be the tuple which we get by applying the four rules exhaustively to the tuple (G, \emptyset, k) . From Lemma 7 we know that this can be done in polynomial time, and that the resulting graph satisfies $|V(G^*) \setminus S| \leq 1$.

Thus $G^* \setminus S$ is either K_1 or the empty graph, and so $G^* \setminus S \in \Pi$ and $pt(G^* \setminus S) = 0, |E(G^* \setminus S)| = 0$. Hence if $k^* \leq 0$ then $(G^* \setminus S, k^*)$ is a YES instance of ABOVE POLJAK-TURZÍK (II). Since all the four rules are safe—Lemmas 13, 17, 18, and 24—we get that in this case (G, k) is a YES instance, and we can return YES. On the other hand if $k^* > 0$ then we know—using Lemma 25—that $|S| < 3k/\lambda' = 6k/(1-\lambda)$, and—from Lemma 8—that $G \setminus S$ is a forest of cliques. This completes the proof. \blacktriangleleft

Proof (of Theorem 1). From Lemma 6 we know that in polynomial time we can either answer YES, or find a set S such that $|S| \leq \frac{6}{1-\lambda}k$ and $G \setminus S$ is a forest of cliques. In the latter case we have reduced the original problem instance to an instance of STRUCTURED ABOVE POLJAK-TURZÍK (II) (See Definition 5). The theorem follows since—by assumption—this latter problem is FPT. \blacktriangleleft

4 Applications

In this section we use Theorem 1 to show that ABOVE POLJAK-TURZÍK (II) is FPT for almost all natural examples of λ -extendible properties listed by Poljak and Turzík [18]. For want of space, we defer the definitions and all proofs to Appendix B.

4.1 Application to Partitioning Problems

First we focus on properties specified by a homomorphism to a vertex transitive graph. As a graph is h -colorable if and only if it has a homomorphism to K_h , searching for a maximal h -colorable subgraph is one of the problems resolved in this section. In particular, a maximum cut equals a maximum bipartite subgraph and, hence, is also one of the properties studied in this section. We use \mathcal{G} to denote the class of graphs—oriented or edge-labelled—to which the property in question belongs.

It is not difficult to see that every vertex-transitive graph G is a regular graph. In particular, if \mathcal{G} allows labels and/or orientations, then for every label and every orientation each vertex of a vertex transitive graph is incident to the same number of edges of the given label and the given orientation.

► **Lemma 26.** *[*] Let G_0 be a vertex-transitive graph with at least one edge of every label and orientation allowed in \mathcal{G} . Then the property “to have a homomorphism to G_0 ” is strongly d/n_0 -extendible in \mathcal{G} , where n_0 is the number of vertices of G_0 and d is the minimum number of edges of the given label and the given orientation incident to any vertex of G_0 over all labels and orientations allowed in \mathcal{G} .*

Note that while the above lemma poses no restrictions on the graphs considered, we can prove the following only for simple graphs.

► **Lemma 27.** *[*] If G_0 is an unoriented unlabeled graph, then the property “to have a homomorphism into G_0 ” is FPT on almost-forests of cliques.*

Lemma 26 and Lemma 27, together with Theorem 1 immediately imply the following corollary.

► **Corollary 28.** *The problem APT (“to have a homomorphism into G_0 ”) is fixed-parameter tractable for every unoriented unlabeled vertex transitive graph G_0 .*

In particular, by setting $G_0 = K_q$ we get the following result.

► **Corollary 29.** *Given a graph G with m edges and n vertices and an integer k , it is FPT to decide, whether G has an q -colorable subgraph with at least $m \cdot (q - 1)/q + (n - 1)/(2q) + k$ edges.*

This shows that the MAX q -COLORABLE SUBGRAPH problem is FPT when parameterized above the Poljak and Turzík bound [18].

4.2 Finding Acyclic Subgraphs of Oriented Graphs

In this section we show how to apply our result to the problem of finding a maximum-size directed acyclic subgraph of an oriented graph, where the size of the subgraph is defined as the number of arcs in the subgraph. Recall that an oriented graph is a directed graph where between any two vertices there is at most one arc. We show that Theorem 1 applies to this problem. To this end we need the following two lemmata.

► **Lemma 30.** $[\star]$ *The property Π : “acyclic oriented graphs” is strongly $1/2$ -extendible in the class of oriented graphs.*

► **Lemma 31.** $[\star]$ *The property “acyclic oriented graphs” is FPT on almost-forests of cliques.*

Combining Lemmata 30 and 31 with Theorem 1 we get the following corollary.

► **Corollary 32.** *The problem $APT(\text{“acyclic oriented graphs”})$ is fixed-parameter tractable.*

To put this result in some context, we recall a couple of open problems posed by Raman and Saurabh [19]: Are the following questions FPT parameterized by k ?

- Given an oriented directed graph on n vertices and m arcs, does it have a subset of at least $m/2 + 1/2(\lceil n - 1/2 \rceil) + k$ arcs that induces an acyclic subgraph?
- Given a directed graph on n vertices and m arcs, does it have a subset of at least $m/2 + k$ arcs that induces an acyclic subgraph?

In the first question, a “more correct” lower bound is the one of Poljak and Turzík, i.e., $m/2 + 1/2(n - 1)/2$, and the lower bound is true only for connected graphs. Corollary 32 answers the corrected question. Without the connectivity requirement, one can show by adding sufficient number of disjoint oriented 3-cycles that the problem is NP-hard already for $k = 0$.

For the second question, observe that each maximal acyclic subgraph contains exactly one arc from every pair of opposite arcs. Hence we can remove these pairs from the digraph without changing the relative solution size, as exactly half of the removed arcs can be added to any solution to the modified instance. Thus, we can restrict ourselves to oriented graphs.

Now suppose that the oriented graph we are facing is disconnected. It is easy to check that picking two vertices from different connected components and identifying them does not change the solution size, as this way we never create a cycle from an acyclic graph. After applying this reduction rule exhaustively, the digraph becomes an oriented connected graph, and the parameter is unchanged. But then if $k \leq (n - 1)/4$ then $m/2 + k \leq m/2 + (n - 1)/4$ and we can answer YES due to Corollary 4. Otherwise $n \leq 4k$, we have a linear vertex kernel, and we can solve the problem by the well known dynamic programming on the kernel [20]. The total running time of this algorithm is $O(2^{4k} \cdot k^2 + m)$. The smallest kernel previously known for this problem is by Gutin et al., and has a quadratic number of arcs [13].

5 Conclusion and Open Problems

In this paper we studied a generalization of the graph property of being bipartite, from the point of view of parameterized algorithms. We showed that for every strongly λ -extendible property Π which satisfies an additional “solvability” constraint, the ABOVE POLJAK-TURZÍK (Π) problem is FPT. As an illustration of the usefulness of this result, we obtained FPT algorithms for the above-guarantee versions of three graph problems.

Note that for each of the three problems—MAX-CUT, MAX q -COLORABLE SUBGRAPH, and ORIENTED MAX ACYCLIC DIGRAPH—for which we used Theorem 1 to derive FPT algorithms for the above-guarantee question, we needed to devise a separate FPT algorithm which works for graphs that are at a vertex deletion distance of $O(k)$ from forests of cliques. We leave open the important question of finding a right logic that captures these examples, and of showing that any problem expressible in this logic is FPT parameterized by deletion distance to forests of cliques. We also leave open the kernelization complexity question for λ -extendible properties.

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A

 Deferred Proofs

► **Lemma 7.** *Rules 1 to 4 can be exhaustively applied to an instance (G, k) of ABOVE POLJAK-TURZÍK (Π) in polynomial time. The resulting tuple (G^*, S, k^*) has $|V(G^*) \setminus S| \leq 1$.*

Proof. Let $(\tilde{G}, \tilde{S}, \tilde{k}), (G', S', k')$ be as in the description above. It is not difficult to verify that (i) each rule can be applied once in polynomial time, (ii) for each application of a rule, if $\tilde{G} \setminus \tilde{S}$ is connected then so also is $G' \setminus S'$, and (iii) each rule strictly reduces the size of the graph $\tilde{G} \setminus \tilde{S}$ —either by deleting vertices from \tilde{G} , or by adding vertices to \tilde{S} . Crowston et al. have shown [5, Lemma 7] that if $\tilde{G} \setminus \tilde{S}$ is a connected graph with at least two vertices, then at least one of these four rules apply to the tuple $(\tilde{G}, \tilde{S}, \tilde{k})$. Since none of the conditions for applying a reduction rule depends on the value of \tilde{k} , and since the only difference between our set of rules and theirs is the way in which \tilde{k} is modified, their result implies this lemma. ◀

► **Lemma 10.** *Let v be a cut vertex of a connected graph G , and let $\mathcal{C} = \{C_1, C_2, \dots, C_r\}; r \geq 2$ be a family of sets of vertices of G such that for every $i \neq j$ we have $C_i \cap C_j = \{v\}$, there is no edge between $C_i \setminus \{v\}$ and $C_j \setminus \{v\}$ and $\bigcup_{1 \leq i \leq r} C_i = V(G)$. For $1 \leq i \leq r$, let $H_i \in \Pi$ be a subgraph of $G[C_i]$ with $pt(G[C_i]) + k_i$ edges, and let $H = (V(G), \bigcup_{i=1}^r E(H_i))$. Then H is a subgraph of G , $H \in \Pi$, and $|E(H)| \geq pt(G) + \sum_{i=1}^r k_i$.*

Proof. Since there are no edges between $C_i \setminus \{v\}$ and $C_j \setminus \{v\}$ for $i \neq j$, and $\bigcup_{1 \leq i \leq r} C_i = V(G)$, every edge of G is in exactly one $G[C_i]$. Therefore, H is a subgraph of G . Also as v is a cut vertex in G , it is a cut vertex in H and the blocks of H are exactly the blocks of H_i 's. Since each H_i is in Π it follows from the block additivity property of Π that $H \in \Pi$.

Since $pt(G[C_i]) = \lambda e_G(C_i) + \lambda'(|C_i| - 1)$, we get

$$\begin{aligned} |E(H)| &= \sum_{i=1}^r |E(H_i)| = \sum_{i=1}^r (pt(G[C_i]) + k_i) = \lambda \sum_{i=1}^r e_G(C_i) + \lambda' \sum_{i=1}^r |C_i| - 1 + \sum_{i=1}^r k_i \\ &= \lambda |E(G)| + \lambda'(|V(G)| - 1) + \sum_{i=1}^r k_i = pt(G) + \sum_{i=1}^r k_i. \end{aligned}$$

◀

► **Lemma 11.** *Let G be a graph, and let $S \subseteq V(G)$ be such that there exists a subgraph $H_S \in \Pi$ of $G[S]$ with at least $pt(G[S]) + \lambda' + k_S$ edges, and a subgraph $\overline{H} \in \Pi$ of $G \setminus S$ with at least $pt(G \setminus S) + \lambda' + \overline{k}$ edges. Then there is a subgraph $H \in \Pi$ of G with at least $pt(G) + \lambda' + k_S + \overline{k}$ edges.*

Proof. Let $F = \delta(S)$, and consider the subgraph $G' = (V(G), E(H_S) \cup E(\overline{H}) \cup F)$. Observe that $G'[S] = H_S \in \Pi$, and $G' \setminus S = \overline{H} \in \Pi$. Thus the strong λ -subgraph extension property of Π applies to the pair (G', S) , and for the weight function which assigns unit weights to all edges in G' , we get that the graph G' has a spanning subgraph $H \in \Pi$ which contains all the edges in $E(H_S) \cup E(\overline{H})$ and at least a λ -fraction of the edges in F . Thus

$$\begin{aligned} |E(H)| &\geq |E(H_S)| + |E(\overline{H})| + \lambda|F| \\ &\geq pt(G[S]) + \lambda' + k_S + pt(G \setminus S) + \lambda' + \overline{k} + \lambda|F| \\ &= \lambda(|E(G[S])| + |E(G \setminus S)| + |F|) + \lambda'(|S| + |V(G) \setminus S|) + k_S + \overline{k} \\ &= \lambda|E(G)| + \lambda'|V(G)| + k_S + \overline{k} = pt(G) + \lambda' + k_S + \overline{k} \end{aligned}$$

◀

► **Lemma 16.** *Let G be a graph formed from the graph K_4 —also with any kind of orientation or labelling—by removing one edge. Then (i) $\text{val}(G) \geq 3$, (ii) $\text{val}(G) \geq 4$ if $\lambda > 1/3$, and (iii) $\text{val}(G) = 5$ if $\lambda > 1/2$. As a consequence,*

$$\text{val}(G) \geq \text{pt}(G) + \lambda' + \begin{cases} (1 - 3\lambda) & \text{if } \lambda \leq 1/3, \\ (2 - 3\lambda) & \text{if } 1/3 < \lambda \leq 1/2, \text{ and,} \\ (3 - 3\lambda) & \text{if } \lambda > 1/2. \end{cases} \quad (2)$$

Proof. A spanning tree of G has three edges, and so claim (i) follows from Observation 14.

Let $V(G) = \{x, y, u, v\}$, and let $\{x, y\} \notin E(G)$. Consider the vertex subset $S = \{x, v\}$, for which $G[S] = K_2 \in \Pi$, $G \setminus S = G[\{y, u\}] = K_2 \in \Pi$, and $|\delta(S)| = 3$. Applying the strong λ -subgraph extension property—for unit edge weights—on the set S we get that there exists a subgraph $H' \in \Pi$ of G which has at least $2 + 3\lambda$ edges. Since $\lambda > 1/3 \implies 2 + 3\lambda > 3$, we get claim (ii).

Now consider the subgraph $G' = G[\{x, u, v\}]$, and its vertex subset $S' = \{x\}$. We apply the strong λ -subgraph extension property—again for unit edge weights—to the pair (G', S') . Since $G'[S'] = K_1 \in \Pi$, $G' \setminus S' = K_2 \in \Pi$ and $|\delta(S')| = 2$, there exists a subgraph $H'' \in \Pi$ of G' which has at least $1 + 2\lambda$ edges. For $\lambda > 1/2$ this is at least 3 edges, and so in this case $H'' = G'$ and $G[\{x, u, v\}] \in \Pi$. Hence we can use the strong λ -subgraph extension property for G and $S = \{y\}$ to get a subgraph of G with at least $3 + 2\lambda$ edges. For $\lambda > 1/2$ this means all the five edges of G , proving claim (iii).

The second part of the lemma follows from these claims since $2\lambda + 4\lambda' = 2$. ◀

► **Lemma 17.** *Rule 2 is safe.*

Proof. We reuse the notation of the rule. Let X_1, \dots, X_d be the cliques deleted by the rule, and let X_{d+1} be the remaining component (if any) of $\tilde{G} \setminus (\tilde{S} \cup \{v\})$. For $1 \leq i \leq d + 1$ let $C_i = X_i \cup \{v\}$. Since $\tilde{G}[C_{d+1}] = G' \setminus S'$, by assumption we have that $\text{val}(\tilde{G}[C_{d+1}]) = \text{val}(G' \setminus S') \geq \text{pt}(G' \setminus S') + k' = \text{pt}(\tilde{G}[C_{d+1}]) + k'$. As we show below, for $1 \leq i \leq d$ we have that $\text{val}(\tilde{G}[C_i]) \geq \text{pt}(\tilde{G}[C_i]) + \lambda'$. Applying Lemma 10 to the graph $\tilde{G} \setminus \tilde{S}$ and the family $\mathcal{C} = \{C_1, \dots, C_{d+1}\}$, we get $\text{val}(\tilde{G} \setminus \tilde{S}) \geq \text{pt}(\tilde{G} \setminus \tilde{S}) + d\lambda' + k' = \text{pt}(\tilde{G} \setminus \tilde{S}) + \tilde{k}$, where the last equality uses the fact that $k' + d\lambda' = \tilde{k}$.

To complete the proof it is sufficient to show that $\text{val}(\tilde{G}[C_i]) \geq \text{pt}(\tilde{G}[C_i]) + \lambda'$; $1 \leq i \leq d$. Consider a deleted clique X_i . Since (i) $\tilde{G} \setminus \tilde{S}$ is connected, and (ii) Rule 1 does not apply, it follows that there exist $x, y \in X_i$ such that x is adjacent to v and y is not adjacent to v in $\tilde{G} \setminus \tilde{S}$. We now consider two cases.

If $|X_i|$ is even, then consider the vertex subset $T = \{v, x, y\}$. The subgraph $\tilde{G}[T] = K_{2,1} \in \Pi$ of $\tilde{G}[C_i]$ has $\text{pt}(\tilde{G}[T]) + \lambda' + \lambda'$ edges (Observation 15). Since $C_i \setminus T = X_i \setminus \{x, y\}$ is a clique in $\tilde{G}[C_i]$ with an even number of vertices, it contains a perfect matching. Therefore we get from Corollary 12 that the graph $\tilde{G}[C_i]$ has a subgraph $H \in \Pi$ with at least $\text{pt}(\tilde{G}[C_i]) + 2\lambda'$ edges.

If $|X_i|$ is odd, then let $T = \{x, v\}$. Now the subgraph $\tilde{G}[T] = K_2 \in \Pi$ of $\tilde{G}[C_i]$ has at least $\text{pt}(\tilde{G}[T]) + \lambda'$ edges. Also, $C_i \setminus T = X_i \setminus \{x\}$ is a clique in $\tilde{G}[C_i]$ with an even number of vertices and hence has a perfect matching. Once again using Corollary 12, we get that $\tilde{G}[C_i]$ has a subgraph $H \in \Pi$ with at least $\text{pt}(\tilde{G}[C_i]) + \lambda'$ edges. ◀

► **Observation 19.** *Whenever Rule 4 applies, there is exactly one component to be deleted, and this component has at least 2 vertices.*

Proof. Suppose the rule can be applied, and there are at least 2 components to be deleted. Pick two vertices u and v in two such distinct components. If the graph $\tilde{G} \setminus (\tilde{S} \cup \{x\})$ is disconnected, then there is a component C_x of this graph which is not connected to y . Similarly, if $\tilde{G} \setminus (\tilde{S} \cup \{y\})$ is disconnected, then there is a component C_y of this graph which is not connected to x . But if either of these happens, then since neither $C_x \cup \{y\}$ nor $C_y \cup \{x\}$ is a clique, the rule does not apply—a contradiction. Hence we get that the graph $\tilde{G} \setminus (\tilde{S} \cup \{x\})$ is connected. But then so is the graph $\tilde{G} \setminus (\tilde{S} \cup \{u, x, v\})$, and as we have $\{u, x\}, \{x, v\} \in E(\tilde{G})$, $\{u, v\} \notin E(\tilde{G})$, Rule 3 applies—a contradiction. So there is exactly one component to be deleted. Now if the only component to be deleted has only one vertex v , then $\tilde{G} \setminus (\tilde{S} \cup \{x, v, y\})$ is connected, we have $\{x, v\}, \{v, y\} \in E(\tilde{G})$, $\{x, y\} \notin E(\tilde{G})$, and so Rule 3 applies, a contradiction. \blacktriangleleft

► **Lemma 20.** *If none of Rules 1, 2, and 3 applies to $(\tilde{G}, \tilde{S}, \tilde{k})$, and Rule 4 does apply, then one can find*

- *A vertex $r \in V(\tilde{G} \setminus \tilde{S})$ and a set $X \subseteq V(\tilde{G} \setminus \tilde{S})$ such that X is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{r\})$, and the graph $(\tilde{G} \setminus \tilde{S})[X \cup \{r\}]$ is 2-connected;*
- *Vertices $x, y \in X$ such that $\{x, y\} \notin E(\tilde{G})$ and*
 - *$(\tilde{G} \setminus \tilde{S}) \setminus \{x, y\}$ has exactly two components G', C ,*
 - *$r \in G'$, and $C \cup \{x\}, C \cup \{y\}$ are cliques, and,*
 - *Each of x, y is adjacent to some vertex in G'*

Proof. Crowston et al. show [5, Lemma 7] that to any connected graph with at least one edge, at least one of Rules 1–4 applies. Our lemma corresponds directly to case 1.(b).iii.C of their case analysis, by setting $a = x, c = y$. For the last point, observe that if one of x, y is not adjacent to any vertex in G' , then Rule 2 would apply. \blacktriangleleft

► **Lemma 21.** *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if x, y are the vertices to be added to \tilde{S} and C the clique to be deleted, then $N(x) \cup N(y) \setminus (C \cup \tilde{S})$ contains at most one vertex z such that $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

Proof. In the application of Rule 4 we set x, y, C as in Lemma 20. Further, let r, X be as in Lemma 20. Then since $x, y \in X$ and X is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{r\})$, we have that $(N(x) \cup N(y)) \subseteq X \cup \{r\}$. Since $(\tilde{G} \setminus \tilde{S})[X \cup \{r\}]$ is 2-connected, it follows that r is the only vertex in $X \cup \{r\}$ which could possibly be a cut vertex of $(\tilde{G} \setminus \tilde{S})$. \blacktriangleleft

► **Lemma 22.** *Whenever Rule 4 applies, with x, y the vertices to be added to \tilde{S} and C the clique to be deleted, every u in $N(x) \setminus (C \cup \tilde{S})$ is a cut vertex in $\tilde{G} \setminus (\tilde{S} \cup \{x\})$ and every u in $N(y) \setminus (C \cup \tilde{S})$ is a cut vertex in $\tilde{G} \setminus (\tilde{S} \cup \{y\})$.*

Proof. We only prove the first part, and the second part follows by symmetry. Assume that for some $u \in N(x) \setminus (C \cup \tilde{S})$ the graph $\tilde{G} \setminus (\tilde{S} \cup \{x, u\})$ is connected, and let w be a vertex of C . Since $|C| \geq 2$, the graph $\tilde{G} \setminus (\tilde{S} \cup \{x, u, w\})$ is also connected and as $\{x, u\}, \{x, w\} \in E$ and $\{u, w\} \notin E$, Rule 3 applies to $\tilde{G} \setminus \tilde{S}$ —a contradiction. Hence $\tilde{G} \setminus (\tilde{S} \cup \{x, u\})$ is disconnected for every $u \in N(x) \setminus (C \cup \tilde{S})$. \blacktriangleleft

► **Lemma 23.** *Suppose Rules 1, 2, and 3 do not apply, and Rule 4 applies. Then we can apply Rule 4 in such a way that if x, y are the vertices to be added to \tilde{S} and C the clique to be deleted, then $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{z\}$, and $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected.*

Proof. In the application of Rule 4 we set x, y, C as in Lemma 20. Then Lemma 21 applies to this application. Let G' be as in Lemma 20. Then $G' = \tilde{G} \setminus (\tilde{S} \cup C \cup \{x, y\})$, and from the last point of Lemma 20 we get that $N(x) \setminus (C \cup \tilde{S}) \neq \emptyset$ and $N(y) \setminus (C \cup \tilde{S}) \neq \emptyset$.

First, observe that if $N(x) \setminus (C \cup \tilde{S}) = \{z\}$, then $\tilde{G} \setminus (\tilde{S} \cup \{x, z\})$ is disconnected only if $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ is disconnected and so from Lemma 22 we get that z is a cut vertex of $\tilde{G} \setminus \tilde{S}$. By a similar argument, if $N(y) \setminus (C \cup \tilde{S}) = \{z\}$ then z is a cut vertex of $\tilde{G} \setminus \tilde{S}$. Now if $|N(x) \setminus (C \cup \tilde{S})| = |N(y) \setminus (C \cup \tilde{S})| = 1$ and $N(x) \setminus (C \cup \tilde{S}) \neq N(y) \setminus (C \cup \tilde{S})$, then we have two different cut vertices of $\tilde{G} \setminus \tilde{S}$ adjacent to vertices x and y , contradicting Lemma 21. So if $|N(x) \setminus (C \cup \tilde{S})| = |N(y) \setminus (C \cup \tilde{S})| = 1$ then there is nothing more to prove.

Next we consider the case $|N(x) \setminus (C \cup \tilde{S})| \geq 2$. Let $Z = N(x) \setminus (C \cup \tilde{S})$, $G_x = \tilde{G} \setminus (\tilde{S} \cup \{x\})$. We claim that there exist two vertices $a_1 \neq a_2 \in Z$ and two vertex subsets $A_1, A_2 \subseteq V(G_x)$ such that (i) A_1 is a connected component of $G_x \setminus \{a_1\}$, (ii) A_2 is a connected component of $G_x \setminus \{a_2\}$, and (iii) neither A_1 nor A_2 contains a vertex of Z . To see this, recall that by Lemma 22 each vertex in Z is a cut vertex of G_x . Hence each vertex in Z is an internal node in the block graph B of G_x . Root the tree B at an arbitrary internal node, and mark all the internal nodes which are in Z . Say that an internal node $u \in Z$ of B is *good* if there is at least one subtree T of B rooted at a child node of u such that no node of T is marked. Consider the operation of repeatedly deleting unmarked leaves from B . Exhaustively applying this operation results in a subtree of B whose leaves are all good nodes in Z . Since we started with at least two marked nodes, we end up with at least two good nodes. Let a_1, a_2 be two of these good nodes. For $i \in \{1, 2\}$, let A_i denote the subgraph of G_x represented by a subtree rooted at some child node of a_i . Then a_1, a_2, A_1, A_2 satisfy the claim.

By Lemma 21 at least one of $\{a_1, a_2\}$, say a_1 , is *not* a cut vertex of $\tilde{G} \setminus \tilde{S}$. Since a_1 is a cut vertex of $G_x = \tilde{G} \setminus (\tilde{S} \cup \{x\})$ and A_1 is a component of $G_x \setminus \{a_1\}$, we get that in the graph $\tilde{G} \setminus \tilde{S}$ there is an edge from the vertex x to some vertex in A_1 . As $Z \cap A_1 = \emptyset$, this implies $A_1 \cap C \neq \emptyset$, from which it follows—since deleting vertex a_1 does not affect the connectedness of $\tilde{G}[C \cup \{y\}]$ —that $C \cup \{y\} \subseteq A_1$. Then $N(y) \subseteq A_1 \cup \{a_1\}$ and, in particular, y is not adjacent to a_2 . Also, since $A_1 \cap A_2 = \emptyset$ by construction, we get that $A_2 \cap (C \cup \{y\}) = \emptyset$. Since—again by construction— $A_2 \cap Z = \emptyset$, we have that $N(x) \cap A_2 = \emptyset$. From this and from the fact that A_2 is a connected component of $G_x \setminus \{a_2\}$, we get that A_2 is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{a_2\})$. Thus a_2 is a cut vertex of $\tilde{G} \setminus \tilde{S}$ which is adjacent to x and not to y .

If $N(y) \setminus (C \cup \tilde{S}) = \{z\}$ then—as shown above— z is a cut vertex of $\tilde{G} \setminus \tilde{S}$ which is adjacent to y . But then z and a_2 are two different cut vertices of $\tilde{G} \setminus \tilde{S}$, both adjacent to x or y , which contradicts Lemma 21. On the other hand, if $|N(y) \setminus (C \cup \tilde{S})| \geq 2$, then we can repeat the above argument to get a cut vertex b_2 of $\tilde{G} \setminus \tilde{S}$ which is adjacent to y and not adjacent to x . Hence $b_2 \neq a_2$ and, again, we get a contradiction with Lemma 21. Therefore, indeed $N(x) \setminus (C \cup \tilde{S}) = N(y) \setminus (C \cup \tilde{S}) = \{z\}$ and z is a cut vertex in $\tilde{G} \setminus \tilde{S}$. ◀

► **Lemma 24.** *Rule 4 is safe.*

Proof. We follow the notation used in the rule. We assume—as for all safeness proofs—that $\text{val}(G' \setminus S') \geq \text{pt}(G' \setminus S') + k'$, and prove that $\text{val}(\tilde{G} \setminus \tilde{S}) \geq \text{pt}(\tilde{G} \setminus \tilde{S}) + \tilde{k}$. Recall that for this rule $\tilde{k} = k' + \lambda'$. By Observation 19 there is exactly one component C_i which satisfies condition (2) of the rule and which is removed by the rule. Further, $C = V(C_i)$ is a clique with at least 2 vertices. Let $u, v \in C$.

If $|C|$ is odd, then consider the vertex subset $T = \{x, u, y\}$. The subgraph $\tilde{G}[T] = K_{2,1} \in \Pi$ has $\text{pt}(\tilde{G}[T]) + \lambda' + \lambda'$ edges (Observation 15). Since $C \setminus \{u\}$ is a clique with an even number of vertices, it has a perfect matching. So we get from Corollary 12 that the graph $\tilde{G}[C \cup \{x, y\}]$

has a subgraph $H \in \Pi$ with at least $pt(\tilde{G}[C \cup \{x, y\}]) + \lambda' + \lambda'$ edges. Observe that $G' \setminus S' = (\tilde{G} \setminus \tilde{S}) \setminus (C \cup \{x, y\})$. Applying Lemma 11 to the graph $\tilde{G} \setminus \tilde{S}$ and its vertex subset $C \cup \{x, y\}$ we get $val(\tilde{G} \setminus \tilde{S}) \geq pt(\tilde{G} \setminus \tilde{S}) + \lambda' + k' = pt(\tilde{G} \setminus \tilde{S}) + \tilde{k}$, as required.

If $|C|$ is even, then let $T = \{x, y, u, v\}$. The subgraph $\tilde{G}[T]$ is a graph formed from K_4 by removing an edge, and Lemma 16 gives λ -dependent lower bounds for $val(\tilde{G}[T])$. Applying Corollary 12 to the graph $\tilde{G}[C \cup \{x, y\}]$ and its subgraphs $\tilde{G}[T]$ and $\tilde{G}[C \setminus \{u, v\}]$ —which forms a clique on an even number of vertices and thus has a perfect matching—we get the following lower bounds:

$$val(\tilde{G}[C \cup \{x, y\}]) \geq pt(\tilde{G}[C \cup \{x, y\}]) + \lambda' + \begin{cases} (1 - 3\lambda) & \text{if } \lambda \leq 1/3, \\ (2 - 3\lambda) & \text{if } 1/3 < \lambda \leq 1/2, \text{ and,} \\ (3 - 3\lambda) & \text{if } \lambda > 1/2. \end{cases} \quad (3)$$

By Lemma 23 we can assume that there is a vertex z in $\tilde{G} \setminus \tilde{S}$ such that $C \cup \{x, y\}$ is a connected component of $\tilde{G} \setminus (\tilde{S} \cup \{z\})$ and z is adjacent to both x and y . We now apply the strong λ -subgraph extension property of Π to the subgraph $\tilde{G}[C \cup \{x, y, z\}]$ and the subset $\{z\}$. Since there are exactly two edges from z to $\tilde{G}[C \cup \{x, y, z\}]$, we gain at least 2λ edges in this process. Note that this implies a gain of *both* the edges if $\lambda > 1/2$, and at least one edge otherwise. From this and using the fact that $pt(\tilde{G}[C \cup \{x, y, z\}]) = pt(\tilde{G}[C \cup \{x, y\}]) + 2\lambda + \lambda'$ we get from Equation 3 that $val(\tilde{G}[C \cup \{x, y, z\}]) \geq pt(\tilde{G}[C \cup \{x, y\}]) + \lambda'$ for all $0 < \lambda < 1$.

Applying Lemma 10 to the graph $\tilde{G} \setminus \tilde{S}$, cut vertex z and vertex subsets $V(\tilde{G}) \setminus (\tilde{S} \cup C \cup \{x, y\} = V(G' \setminus S'))$ and $C \cup \{x, y, z\}$, we get $val(\tilde{G} \setminus \tilde{S}) \geq pt(\tilde{G} \setminus \tilde{S}) + k' + \lambda' = pt(\tilde{G} \setminus \tilde{S}) + \tilde{k}$. ◀

► **Lemma 25.** *Let \tilde{G} be a connected graph, $\tilde{S} \subseteq V(\tilde{G})$, and $\tilde{k} \in \mathbb{N}$, and let one application of Rule 1, 2, 3, or 4 to $(\tilde{G}, \tilde{S}, \tilde{k})$ result in the tuple (G', S', k') . Then $|S' \setminus \tilde{S}| \leq 3(\tilde{k} - k')/\lambda'$.*

Proof. We distinguish the rule applied. For Rule 1, $\tilde{S} = S'$ and $\tilde{k} = k'$. For Rule 2 we have $|S' \setminus \tilde{S}| = 1$, while $\tilde{k} - k' \geq \lambda'$. Hence $|S' \setminus \tilde{S}| \leq (\tilde{k} - k') \cdot 1/\lambda'$. Similarly, for Rule 3 we have $|S' \setminus \tilde{S}| = 3$, $\tilde{k} - k' = \lambda'$, and $|S' \setminus \tilde{S}| \leq (\tilde{k} - k') \cdot 3/\lambda'$. Finally, for Rule 4 we have $|S' \setminus \tilde{S}| = 2$ and $\tilde{k} - k' = \lambda'$, and $|S' \setminus \tilde{S}| \leq (\tilde{k} - k') \cdot 2/\lambda'$. ◀

B Applications

► **Definition 33** (Graph homomorphisms). A homomorphism from a graph G to a graph H is a mapping $\phi : V(G) \rightarrow V(H)$ such that for each edge $\{u, v\} \in E(G)$ the pair $\{\phi(u), \phi(v)\}$ is an edge in H , if $\{u, v\}$ has a label, then $\{\phi(u), \phi(v)\}$ has the same label and if (u, v) is an oriented edge of G , then $(\phi(u), \phi(v))$ is an oriented edge of H . The set of all homomorphisms from G to H will be denoted $\text{HOM}(G, H)$, and $\text{hom}(G, H) = |\text{HOM}(G, H)|$.

► **Definition 34** (Graph automorphisms and vertex-transitive graphs). For a graph G , a bijection $\phi : V(G) \rightarrow V(G)$ is an *automorphism* of G if it is a homomorphism from G to itself. A graph G is *vertex-transitive* if for any two vertices $u, v \in V(G)$ there is an automorphism ϕ of G such that $\phi(u) = v$.

► **Lemma 26.** *Let G_0 be a vertex-transitive graph with at least one edge of every label and orientation allowed in \mathcal{G} . Then the property “to have a homomorphism to G_0 ” is strongly d/n_0 -extendible in \mathcal{G} , where n_0 is the number of vertices of G_0 and d is the minimum number of edges of the given label and the given orientation incident to any vertex of G_0 over all labels and orientations allowed in \mathcal{G} .*

Proof. Let $\mathcal{H} \subseteq \mathcal{G}$ be the set of graphs which have a homomorphism to G_0 . We show that the set \mathcal{H} satisfies all the three requirements for being strongly d/n_0 -extendible.

A map which takes the single vertex in K_1 to any vertex of G_0 is a homomorphism from K_1 to G_0 . Let G be K_2 possibly with some orientation and label and (u_0, v_0) be an edge in G_0 of the same orientation and label. A map which takes the two vertices in G to u_0, v_0 , respectively, is a homomorphism from G to G_0 . Thus both K_1 and K_2 with all orientations and labels are in \mathcal{H} .

Lemma 36 shows that \mathcal{H} has the block additivity property, and from Lemma 38 we get that \mathcal{H} has the strong λ -subgraph extension property for $\lambda = d/n_0$. ◀

► **Observation 35.** Let G, H be two graphs such that there is a homomorphism ϕ from G to H , and (ii) H is vertex-transitive. Then for any two vertices $u \in V(G), v \in V(H)$, there is a homomorphism φ from G to H which maps u to v .

Proof. Let $\phi(u) = x$, and let θ be an automorphism of H such that $\theta(x) = v$. Since H is vertex-transitive, such an automorphism exists. Set $\varphi := \theta \circ \phi$. ◀

► **Lemma 36.** Let G_0 be a vertex-transitive graph. Then the property “to have a homomorphism to G_0 ” has the block-additivity property.

Proof. Let \mathcal{H} be the set of graphs which have a homomorphism to G_0 . Let H be a graph in \mathcal{H} , and let ϕ be a homomorphism from H to G_0 . Let H_i be a block of H . Observe that any edge (u, v) in H_i is present in H as well, and therefore $(\phi(u), \phi(v))$ is an edge in G_0 . Thus ϕ restricted to H_i —in the natural way—is a homomorphism from H_i to G_0 , and so H_i is in \mathcal{H} .

For the converse, let $\{H_i \mid 1 \leq i \leq t\}$ be the blocks of a graph H , and let each H_i be in \mathcal{H} . Then there is a homomorphism from each graph H_i to the graph G_0 . We now show how to construct a homomorphism from H to G_0 . We assume—without loss of generality; see below—that the graph H is connected.

Recall that the vertex set of the *block graph* T_H of H consists of the blocks and cut vertices of H , and that a block B and a cut vertex c of H are adjacent in T_H exactly when c is a vertex in B . We root the tree T_H at some (arbitrary) cut vertex r of H . Each level of T_H then consists entirely of either cut vertices or blocks. We now define a mapping φ from H to G_0 by starting from the root of the block graph T_H , and going down level by level.

We set $\varphi(r)$ to be some arbitrary vertex of G_0 . We now consider each level L in T_H which consists entirely of blocks, in increasing order of levels. For each block H_i in L , we do the following. Let c be the cut vertex which is the parent of H_i in T_H . Note that $\varphi(c)$ has already been defined; let $\varphi(c) = d$. Let ϕ_i be a homomorphism from H_i to G_0 which maps c to the vertex d ; Observation 35 guarantees that such a homomorphism exists. For each vertex x of H_i , we set $\varphi(x) = \phi_i(x)$.

Consider a cut vertex v of H . The above procedure maps v to some vertex of G_0 *exactly once*: If $v = r$, then this mapping is done explicitly at the very beginning of the procedure; otherwise, this is done when the procedure assigns images for the vertices in the *unique* parent block H_i of v . Now consider a vertex v of H which is *not* a cut vertex. The procedure maps v to some vertex of G_0 exactly once, when it assigns images to the unique block to which v belongs. Thus the map φ is a function.

Since no edge of H appears—by definition—in two different blocks of H , and since the mapping for each block is a homomorphism to G_0 , it follows that φ is a homomorphism from H to G_0 . If H is not connected, then we apply this procedure separately to each connected component of H , and this yields a homomorphism from H to G_0 . This completes the proof. ◀

► **Lemma 37.** *Let G_0 be a vertex-transitive graph, and let G be any graph. For vertices u in G and x in G_0 , let $\text{HOM}(G, G_0, u, x)$ denote the set of all homomorphisms from G to G_0 which map u to x , and let $\text{hom}(G, G_0, u, x) = |\text{HOM}(G, G_0, u, x)|$. Then for any vertex u in G and any two vertices x_0, y_0 in G_0 , $\text{hom}(G, G_0, u, x_0) = \text{hom}(G, G_0, u, y_0)$.*

Proof. Let ϕ be an automorphism of G_0 which takes x_0 to y_0 . The automorphism ϕ defines a map from $\text{HOM}(G, G_0, u, x_0)$ to $\text{HOM}(G, G_0, u, y_0)$, which takes $\varphi \in \text{HOM}(G, G_0, u, x_0)$ to $\phi \circ \varphi \in \text{HOM}(G, G_0, u, y_0)$. This map is one-one: if $\varphi_1, \varphi_2 \in \text{HOM}(G, G_0, u, x_0)$, then $\phi \circ \varphi_1 = \phi \circ \varphi_2 \implies \phi^{-1} \circ \phi \circ \varphi_1 = \phi^{-1} \circ \phi \circ \varphi_2 \implies \varphi_1 = \varphi_2$. In a similar fashion, the inverse automorphism ϕ^{-1} defines a one-one map from $\text{HOM}(G, G_0, u, y_0)$ to $\text{HOM}(G, G_0, u, x_0)$. It follows that $\text{hom}(G, G_0, u, x_0) = \text{hom}(G, G_0, u, y_0)$. ◀

► **Lemma 38.** *Let G_0 be a vertex-transitive graph on n_0 vertices and d be the minimum number of edges of the given label and the given orientation incident to any vertex of G_0 over all labels and orientations allowed in \mathcal{G} . Then the property Π : “to have a homomorphism to G_0 ” has the strong λ -subgraph extension property in \mathcal{G} for $\lambda = d/n_0$.*

Proof. This proof is based on a similar argument by Poljak and Turzík [18, Theorem 2].

Let \mathcal{H} be the set of graphs which have a homomorphism to G_0 . Let G be a graph, let $c : E(G) \rightarrow \mathbb{R}_+$ be a weight function, and let $S \subseteq V(G)$ be such that $G[S] \in \mathcal{H}$ and $G \setminus S \in \mathcal{H}$. Let $\delta(S)$ denote the set of edges in G which have exactly one end-point in S , and let w be the sum of the weights of the edges in $\delta(S)$. Let ϕ be a homomorphism from $G \setminus S$ to G_0 . Call a mapping $\varphi : V(G) \rightarrow V(G_0)$ a *proper extension* of ϕ if $\varphi|_S$ is a homomorphism from $G[S]$ to G_0 and $\varphi|_{(V(G)-S)}$ is identical to ϕ . Observe that φ need *not* be a homomorphism from G to G_0 . (Note that the number of proper extensions of ϕ is equal to $\text{hom}(G[S], G_0)$.)

Consider an edge $\{x, u\}$ of $V(G) \setminus S$ with $u \in S$ and $\phi(x) = x_0$. There are exactly $\text{hom}(G[S], G_0) \cdot d/n_0$ proper extensions φ of ϕ such that $\{\varphi(x), \varphi(u)\}$ is an edge of G_0 with the same label and orientation: the vertex x_0 is incident to exactly d such edges in G_0 , and from Lemma 37 there are exactly $\text{hom}(G[S], G_0)/n_0$ homomorphisms from $G[S]$ to G_0 which map the vertex u to any given neighbor of x_0 . From an easy averaging argument, it follows that there exists a proper extension φ of ϕ and a subset $F \subseteq \delta(S)$ of edges with total weight at least dw/n_0 such that φ maps each edge in F to an edge in G_0 with the same label and orientation. This completes the proof. ◀

► **Lemma 27.** *If G_0 is an unoriented unlabeled graph, then the property “to have a homomorphism into G_0 ” is FPT on almost-forests of cliques.*

Proof. Let G be an unlabeled unoriented graph, k an integer, and S a set of vertices of G such that $|S| \leq qk$ and $G \setminus S$ is a forest of cliques. For every mapping $\varphi : S \rightarrow V(G_0)$ the algorithm proceeds as follows. We want to count the number r_φ of edges a subgraph of G homomorphic to G_0 with the homomorphism extending φ can have. Denote by $e_\varphi(S)$ the number of edges $\{u, v\}$ in $E(G(S))$ such that $\{\varphi(u), \varphi(v)\}$ is an edge of G_0 .

We use a table Tab to store for each vertex v of $G \setminus S$ and for each vertex v_0 of G_0 roughly speaking how many edges we could get into the constructed subgraph, if the vertex v was mapped to the vertex v_0 . We initialize the tables by setting $\text{Tab}[v, v_0] = |\{u \in S \mid \{\varphi(u), v_0\} \in E(G_0)\}|$, $G' = G \setminus S$ and $r_\varphi = e_\varphi(S)$. Our aim is to remove the leaf cliques of G' one by one (except possibly for the cut vertex also contained in other cliques) as long as the graph G' is non-empty. The edges incident to deleted vertices are captured either by increasing r_φ if the clique was a connected component of G' or by updating the table of the cut vertex, which separates the clique from the rest of its component.

Let C be leaf clique of G' and let us first assume that C forms a connected component of G' . Next we guess how many vertices of the clique are mapped to individual vertices of G_0 . For a vertex $u \in V(G_0)$ we denote this number n_u . Thus for every $|V(G_0)|$ -tuple of numbers $(n_u)_{u \in V(G_0)}$ such that $\sum_{u \in V(G_0)} n_u = |C|$ we continue as follows. Based on the numbers n_u we compute the number of edges inside C that we get as $\sum_{\{u,v\} \in E(G_0)} n_u \cdot n_v$. It remains to maximize the number of edges we get between C and S . For that purpose consider an auxiliary edge-weighted complete bipartite graph B with one partition formed by C and the other partition being $|C|$ many vertices, out of which n_u are labeled with u for every $u \in V(G_0)$. An edge from $v \in C$ to a vertex labeled u is assigned the weight $\text{Tab}[v, u]$. Now every mapping of vertices of C to vertices of G_0 corresponds to a perfect matching in B and vice versa. Moreover, the number of edges between C and S that we can keep if we want to turn such a mapping into a homomorphism is exactly equal to the weight of the corresponding perfect matching. Hence it is enough to compute the maximum weight perfect matching in B . It is well known that this can be done in time polynomial in the size of B which is $2|C|$.

Let us denote t the maximum over all $|V(G_0)|$ -tuples of numbers $(n_u)_{u \in V(G_0)}$ with $\sum_{u \in V(G_0)} n_u = |C|$ of the sum $b + \sum_{\{u,v\} \in E(G_0)} n_u \cdot n_v$, where b is the size of the maximum weight perfect matching for the graph B as computed for the tuple. The algorithm increases r_φ by t and removes the vertices of C from G' . If G' is non-empty, it continues with another leaf clique.

Now let C be a leaf clique, which doesn't form a connected component of G' and let v be the cut vertex which disconnects C from the rest of its component. For every $v_0 \in V(G_0)$ and for every $|V(G_0)|$ -tuple of numbers $(n_u)_{u \in V(G_0)}$ such that $\sum_{u \in V(G_0)} n_u = |C|$ we want to compute how many edges we get, if v is mapped to v_0 and n_u vertices out of $C \setminus \{v\}$ are mapped to u . For that purpose we again use an auxiliary bipartite graph, this time with $|C| - 1$ vertices in each partition.

Let $t(v_0)$ be the maximum over all $|V(G_0)|$ -tuples of numbers $(n_u)_{u \in V(G_0)}$ with $\sum_{u \in V(G_0)} n_u = |C| - 1$ of the sum $b + \sum_{\{u,v\} \in E(G_0)} n_u \cdot n_v + \sum_{\{u,v_0\} \in E(G_0)} n_u$, where b is the size of the maximum weight perfect matching for the graph B as computed for the tuple $(n_u)_{u \in V(G_0)}$ and v_0 , the second term counts the number of edges we got inside $C \setminus \{v\}$ and the last one counts the edges between c and $C \setminus \{v\}$. The algorithm increases $\text{Tab}[v, v_0]$ by $t(v_0)$ for every $v_0 \in V(G_0)$ and removes the vertices of $C \setminus \{v\}$ from G' . If G' is non-empty, it continues with another leaf clique.

Finally, if G' is empty, then r_φ contains the maximum number of edges we get for the initial mapping φ . Then the maximum number r of edges in subgraph of G homomorphic to G_0 is the maximum of r_φ computed by the algorithm taken over all possible mappings $\varphi : S \rightarrow V(G_0)$. It is enough to compare r with $d/n_0 \cdot |E(G)| + (n_0 - d)/2n_0 \cdot (|V(G)| - 1) + k$ and answer accordingly.

It is easy to check that the algorithm is correct and that it works in $O(n_0^{|S|} \cdot |G|^{O(n_0)}) = O((n_0^q)^k \cdot |G|^{O(n_0)})$ time. \blacktriangleleft

► **Lemma 30.** *The property Π : “acyclic oriented graphs” is strongly $1/2$ -extendible in the class of oriented graphs.*

Proof. Obviously, K_1 and both orientations of K_2 are directed acyclic graphs, hence in Π . If an oriented graph is acyclic, then clearly each of its blocks is acyclic. On the other hand, each cycle is within one block of a graph, and hence, if every block is acyclic, then the graph itself is acyclic. Finally, if G and S are such that $G[S]$ is acyclic and $G \setminus S$ then both the graph formed by removing from G all edges oriented from S to $V(G) \setminus S$ and the one formed

by removing edges oriented from $V(G) \setminus S$ to S are acyclic and one of them is removing less than half of the edges between S and $V(G) \setminus S$, finishing the proof. \blacktriangleleft

► **Lemma 31.** *The property “acyclic oriented graphs” is FPT on almost-forests of cliques.*

Proof. Let $G = (V, E)$ be an unlabeled oriented graph, k an integer, and S a set of vertices of G such that $|S| \leq qk$ and $G \setminus S$ is a forest of cliques. Cliques in this case are in fact tournaments. We first show that if any of the tournaments in $G \setminus S$ is very big, then we can answer yes.

Spencer [21] showed that any tournament on n vertices contains a directed acyclic subgraph with at least $\binom{n}{2}/2 + c \cdot n^{3/2}$ arcs for some absolute positive constant c , and claimed that he can achieve $c > 0.15$. Therefore, for every k there is $b_0(k)$ such that $c \cdot b_0(k)^{3/2} \geq b_0(k)/4 + k$ and every tournament on b vertices with $b \geq b_0$ contains a directed acyclic subgraph with at least $\binom{b}{2}/2 + (b-1)/4 + k + 1/4$ arcs. Note that if $k \geq (5/4c)^2$, then it is enough to take $b_0(k) = k$ and hence, we have $b_0(k) = O(k)$.

If C is a set of $b \geq b_0(k)$ vertices inducing a tournament in $G \setminus S$, then $G[C]$ has a directed acyclic subgraph with $e_G(C)/2 + (|C| - 1)/4 + k + 1/4$ arcs, $G \setminus C$ has a directed acyclic subgraph with $e_G(V \setminus C)/2 + (|V| - |C| - 1)/4$ arcs by Corollary 4, and by Lemma 11, G has a directed acyclic subgraph with $|E|/2 + (|V| - 1)/4 + k$ arcs and we can answer yes. Hence, from now on we assume that the largest tournament in $G \setminus S$ has size at most $b_0(k)$.

We want to find a linear order \prec on V which maximizes the number of arcs $(u, v) \in E$ such that $u \prec v$. We say that such arcs are *along the order*. As for a directed acyclic graph there is an order in which all the arcs are along the order, the maximum size of a directed acyclic subgraph can be found in this way. We proceed similarly to the proof of Lemma 27. For every linear order \prec on S the algorithm proceeds as follows. We want to count the maximum number r_{\prec} of arcs that are along any \prec , where \prec is an extension of \prec . Denote by $e_{\prec}(S)$ the number of arcs in $E(G(S))$ that are along \prec .

We use a table Tab to store for each vertex v of $G \setminus S$ and for each extension \prec' of \prec to $S \cup \{v\}$ how many arcs we could get if the constructed order \prec extends \prec' . We initialize the tables by setting $Tab[v, \prec'] = |\{u \in S \mid ((u, v) \in E \wedge u \prec' v) \vee ((v, u) \in E \wedge v \prec' u)\}|$, $G' = G \setminus S$ and $r_{\prec} = e_{\prec}(S)$. Our aim is to remove the leaf cliques of G' one by one (except possibly for the cut vertex also contained in other cliques) as long as the graph G' is non-empty. The arcs incident to deleted vertices are captured either by increasing r_{\prec} if the clique was a connected component of G' or by updating the table of the cut vertex, which separates the clique from the rest of its component.

Let C be leaf clique of G' and let us first assume that C forms a connected component of G' . For every linear order \prec on $C \cup S$ extending \prec we let $t(\prec) = \sum_{v \in C} Tab(v, \prec|_{S \cup \{v\}}) + |\{u, v \in C \mid (u, v) \in E \wedge u \prec v\}|$. Here the first term counts the arcs got by the placement of each individual vertex of C relatively to the vertices of S and the second one counts the arcs along the order inside C . The algorithm increases $r_{\prec'}$ by the maximum $t(\prec)$ over all extensions \prec of \prec to $C \cup S$ and removes the vertices of C from G' . If G' is non-empty, it continues with another leaf clique.

Now let C be a leaf clique, which doesn't form a connected component of G' and let v be the cut vertex which disconnects C from the rest of its component. For every linear order \prec on $C \cup S$ extending \prec we let $t(\prec) = \sum_{u \in C, u \neq v} Tab(u, \prec|_{S \cup \{u\}}) + |\{u, w \in C \mid (u, w) \in E \wedge u \prec w\}|$. For every extension \prec' of \prec to $S \cup \{v\}$ we increase $Tab(v, \prec')$ by $\max_{\prec} t(\prec)$, where the maximum is taken over all \prec extending \prec' to $S \cup C$. Then the algorithm removes the vertices of $C \setminus \{v\}$ from G' and, if G' is non-empty, it continues with another leaf clique.

Finally, if G' is empty, then r_{\leq} contains the maximum number of arcs we get for the initial order \leq . Then the maximum number r of edges in a directed acyclic subgraph of G is the maximum of r_{\leq} computed by the algorithm taken over all possible linear orders \leq on S . It is enough to compare r with $|E(G)|/2 + (|V(G)| - 1)/4 + k$ and answer accordingly.

It is easy to check that the algorithm is correct. As to the running time, there are at most $(qk)! = k^{O(k)}$ linear orders of S . The algorithm tests linear orders for $C \cup S$, but since each clique is of size at most $b_0(k) = O(k)$, there are also at most $k^{O(k)}$ of these. It takes $O(k^2)$ time to process each linear order. The block decomposition of $G \setminus S$ can be found in $O(|V| + |E|)$ time and by keeping the list of leaves and adding the neighboring block of the currently processed leaf to the list if it becomes leaf after removal of the current leaf, it takes $O(|V|)$ time to find the cliques over the whole run of the algorithm. It follows that the algorithm works in $O(k^{O(k)} \cdot |V| + |E|)$ time. ◀